Measure and Integration

Lecture 4

15/11/10

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Suppose $\mathcal{M}$ is a σ-algebra

Claim $\mathcal{M}$ is a monotone class?

Pf (i) let $A_n \in \mathcal{M}$, $A_n \uparrow$, i.e.

$$A_n \subseteq A_{n+1} \quad \forall\, n \geq 1.$$

Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ ($\because$ $\mathcal{M}$ is a σ-algebra)

(ii) let $A_n \in \mathcal{M}$, $A_n \downarrow$, i.e.

$$A_n \supseteq A_{n+1} \quad \forall\, n \geq 1$$

$\Rightarrow$ $A_n^c \in \mathcal{M}$ $\forall\, n \geq 1$ ($\because$ $\mathcal{M}$ is a σ-alg.)

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{M} \quad \left(\because A_n \downarrow \Rightarrow A_n^c \uparrow \right)$$

$$\|$$

$$\Rightarrow \left(\bigcap_{n=1}^{\infty} A_n\right)^c \in \mathcal{M}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}.$$

Hence $\mathcal{M}$ is a monotone class.

②

③

X — Uncountable set

$\mathcal{M} = \{A \subseteq X \mid A \text{ is countable}\}$

<u>Claim</u> $\mathcal{M}$ is a monotone class.

Pf: (i) Let $A_n \in \mathcal{M}$, $A_n \subseteq A_{n+1}$ $\forall n$

$\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$?

<u>Note</u> A_n countable $\forall n$

$\Longrightarrow \bigcup_{n=1}^{\infty} A_n$ is countable

$\Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

(ii) $A_n \in \mathcal{M}$, $A_n \supseteq A_{n+1}\ \forall n$

$$\Longrightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}?$$

$\bigcap_{n=1}^{\infty} A_n$ is countable?

$$\bigcap_{n=1}^{\infty} A_n \subseteq A_n \quad \forall n$$

(A_n countable)

$$\Longrightarrow \bigcap_{n=1}^{\infty} A_n \text{ is countable}$$

$$\Longrightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}.$$

<u>Claim</u> <u>$\mathcal{M}$ is not a σ-algebra</u>

<u>Note</u> (i) X uncountable $\|$

$\Rightarrow X \notin \mathcal{M}$.

(ii) <u>X uncountable $\Rightarrow \exists$</u>

<u>A</u> $\subseteq X$ s.t. neither A $\|$ or A^c is not countable. $\|$

Pf Suppose A is countable

$\Rightarrow A \in \mathcal{M}$

but A^c is not countable

$\Rightarrow A^c \notin \mathcal{M}$.

$\mathcal{C} \subseteq \mathcal{P}(X)$ ⑥

$$\underline{\mathcal{M}(\mathcal{C})} := \bigcap_{\underline{\mathcal{M} \supseteq \mathcal{C}}} \underline{\mathcal{M}}$$

<u>Claim</u> (i) $\mathcal{C} \subseteq \mathcal{M}(\mathcal{C})$ ✓

(ii) $\mathcal{M}(\mathcal{C})$ is a monotone class.

Pf: $A_n \in \mathcal{M}(\mathcal{C})$, $A_n \downarrow$

$\Downarrow$

$A_n \in \mathcal{M} \ \forall n$

$\Downarrow$

$\bigcap_{n=1}^{\infty} A_n \in \mathcal{M} \ \forall \mathcal{M}$

$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}(\mathcal{C})$

(7)

(ii) $A_n \in \mathcal{M}(\mathcal{C}),\ A_n \uparrow$

$\Rightarrow A_n \in \mathcal{M}\ \forall\, \mathcal{M},\ A_n \uparrow$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}\ \forall\ \mathcal{M}$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}(\mathcal{C})$

$\Longrightarrow$ $\mathcal{M}(\mathcal{C})$ is a monotone class

(iii) $\mathcal{M}(\mathcal{C})$ is smallest such that $\underline{\mathcal{C} \subseteq \mathcal{M}(\mathcal{C})}$

$\mathcal{C}$ an algebra

$+$ monotone class

$\Longrightarrow$ $\mathcal{C}$ is a σ-algebra

Pf (i) $\phi, X \in \mathcal{C}$ ✓

(ii) $A \in \mathcal{C} \Longrightarrow A^c \in \mathcal{C}$ ✓

(iii) $A_n \in \mathcal{C} \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$?

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \Big(\underbrace{\bigcup_{i=1}^{n} A_i}_{\in \mathcal{C}} \Big)$$

$\Longrightarrow \in \mathcal{C}$

$\mathcal{C} \subseteq \mathcal{M}(\mathcal{C})$

Also $\mathcal{C} \subseteq \underline{\mathcal{S}(\mathcal{C})}$

is also a monotone class

$\Longrightarrow \mathcal{C} \subseteq \underline{\mathcal{M}(\mathcal{C})} \subseteq \underline{\mathcal{S}(\mathcal{C})}$

<u>Question</u> $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{M}(\mathcal{C})$??

$\underline{\mathcal{S}(\mathcal{C}) = \mathcal{M}(\mathcal{C})}$.

$\mathcal{M}(\mathcal{A})$ is an algebra

$\Longrightarrow$ (also a monotone class)

$\Longrightarrow$ $\mathcal{M}(\mathcal{A})$ is <u>a σ-algebra</u>

But then $\mathcal{A} \subseteq \mathcal{M}(\mathcal{A})$

$\Longrightarrow$ $\mathcal{A} \subseteq \underline{S(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})}$

<u>To show</u> $\mathcal{M}(\mathcal{A})$ is an algebra

when $\mathcal{A}$ is an algebra ??

<u>claim</u> $\mathcal{M}(\mathcal{A})$ is closed under 'complements',

i.e. $\underline{A \in \mathcal{M}(\mathcal{A}) \implies A^c \in \mathcal{M}(\mathcal{A})}$?

<u>Consider</u>

$$\mathcal{B} := \{ E \subseteq X \mid E^c \in \mathcal{M}(\mathcal{A}) \}$$

& $\underline{\mathcal{M}(\mathcal{A}) \subseteq \mathcal{B}}$. ||

<u>Enough to show</u>

(i) $\mathcal{A} \subseteq \mathcal{B}$.

(ii) <u>$\mathcal{B}$ is a monotone class</u>

(12)

(i) Let $\underline{A \in \mathcal{A}} \Longrightarrow A^c \in \mathcal{A}$ ($\because \mathcal{A}$ algebra)

$\Longrightarrow \underline{A^c \in \mathcal{A} \subseteq \mathcal{M}(\mathcal{A})}$

Hence $\Longrightarrow A \in \mathcal{B}$

i.e., $\mathcal{B} \supseteq \mathcal{A}$

(ii) Let $A_n \in \mathcal{B}$ s.t. $A_n \uparrow$.

$$A_n \in \mathcal{B} \Longrightarrow A_n^c \in \mathcal{M}(\mathcal{A})$$

$$\Longrightarrow \bigcap_{n=1}^{\infty} A_n^c \in \mathcal{M}(\mathcal{A}) \quad (\because A_n^c \downarrow)$$

$$\Longrightarrow \left(\bigcup_{n=1}^{\infty} A_n\right)^c \in \mathcal{M}(\mathcal{A})$$

$$\Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}.$$

Let $A_n \in \mathcal{B}$, $A_n \downarrow$ ($A_n^c \uparrow$) (13)

$\underline{A_n \in \mathcal{B}} \Longrightarrow A_n^c \in \mathcal{M}(\mathcal{A}) \ \forall n$

$\Longrightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{M}(\mathcal{A})$

$\Longrightarrow \underline{\left(\bigcap_{n=1}^{\infty} A_n\right)^c} \in \mathcal{M}(\mathcal{A})$

$\Longrightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{B}.$

$\mathcal{M}(\mathcal{A})$ is closed under unions

Fix $F \in \mathcal{M}(\mathcal{A})$,

$$\mathcal{L}(F) := \{E \subseteq X \mid E \cup F \in \mathcal{M}(\mathcal{A})\}$$

To show $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{L}(F)$?

Try to show: (i) $\mathcal{A} \subseteq \mathcal{L}(F)$ $\forall F \in \mathcal{M}(\mathcal{A})$

(ii) $\mathcal{L}(F)$ is a monotone class.

$E_n \in \mathcal{L}(F)$, $E_n \uparrow \Longrightarrow \bigcup_{n\geq 1}^{\infty} E_n \in \mathcal{L}(F)$?

$\Downarrow$

$E_n \cup F \in \mathcal{M}(\mathcal{A}) \Longrightarrow \bigcup_{n=1}^{\infty}(E_n \cup F) \in \mathcal{M}(\mathcal{A})$

$\Longrightarrow \left(\bigcup_{n=1}^{\infty} E_n\right) \cup F \in \mathcal{M}(\mathcal{A})$

<u>Note</u> If $\underline{F \in \mathcal{A}}$, then $\forall$ $\underline{E \in \mathcal{A}}$

$$\underline{E \cup F} \in \mathcal{A} \subseteq \underline{\mathcal{M}(\mathcal{A})}$$

$\Rightarrow$ $E \in \mathcal{L}(F)$ $\forall$ $E \in \mathcal{A}$

$\Rightarrow$ $\mathcal{A} \subseteq \mathcal{L}(F)$ $\forall$ $F \in \mathcal{A}$

$\Rightarrow$ $\underline{\mathcal{M}(\mathcal{A}) \subseteq \mathcal{L}(F)}$ $\forall$ $\underline{F \in \mathcal{A}}$

<u>Note</u>

$$E \in \mathcal{L}(F) \Longleftrightarrow F \in \mathcal{L}(E)$$

$\forall$ $E \in \mathcal{M}(\mathcal{A}) \Rightarrow \underline{E \in \mathcal{L}(F)}$

$\Longleftrightarrow \underline{F} \in \mathcal{L}(E)$

$\Rightarrow$ $\mathcal{A} \subseteq \mathcal{L}(F)$ $\forall$ $F \in \mathcal{M}(\mathcal{A})$

$$\Longrightarrow \quad \mathcal{M}(\mathcal{A}) \subseteq \underline{\mathcal{L}(F)} \ \forall \ F \in \mathcal{M}(\mathcal{A}) \qquad (16$$

X

$\mathcal{C} \subseteq \mathcal{P}(X)$

Semi-algebra $\subseteq$, $\neq$ Algebra $\subseteq$, $\neq$ σ-algebra $\subseteq$, $\neq$ Monotone class

$\mathcal{M}(\mathcal{A}) = \mathcal{S}(\mathcal{A})$

if $\mathcal{A}$ is an algebra.